It can be argued that the stability characteristics of the linear equation (2.47) are very similar to the stability characteristics of the equation of the form given by (2.46). Since the terms Bt and C will give rise to corresponding terms in both numerical and exact solutions which are also linear in $t(\lambda \neq 0)$, we conclude that (2.46) exhibits short-range numerical instability in the neighbourhood of (t_n, y_n), when the corresponding equation (2.45) with $\lambda = f_y(t_n, y_n)$, exhibits numerical instability. Therefore, the stability analysis will be based on the equation

$$y' = f(t, y) \approx \lambda y, \; y(t_0) = y_0 \tag{2.48}$$

where
$$\lambda = \left(\frac{\partial f}{\partial y}\right)_n$$

and it is assumed that $(\partial f/\partial y)$ is relatively invariant in the region of interest. Equation (2.48) has as its solution

$$y(t) = y(t_0) \exp(\lambda(t - t_0))$$

which at $t = t_0 + nh$ becomes

$$y(t_n) = y(t_0)\, e^{\lambda nh} = y_0 (e^{\lambda h})^n$$

A singlestep method when applied to (2.48) will lead to a first order difference equation which has solution of the form

$$y_n = c_1 (E(\lambda h))^n$$

where c_1 is a constant to be determined from the initial condition and $E(\lambda h)$ is an approximation to $e^{\lambda h}$. We call the singlestep method

$$\text{Absolutely stable if } |E(\lambda h)| \leqslant 1$$
$$\text{Relatively stable if } |E(\lambda h)| \leqslant e^{\lambda h}$$

If $\lambda < 0$, the exact solution decreases as t_n increases and the important condition is the absolute stability, since the numerical solution must also decrease with t_n. If Euler's method is used, we obtain

$$y_{n+1} = y_n + hf_n = E(\lambda h)\, y_n$$

where
$$E(\lambda h) = 1 + \lambda h.$$

Obviously, Euler's method is absolutely stable if

$$|1 + \lambda h| \leqslant 1 \text{ or } -2 \leqslant \lambda h < 0$$

If $\lambda > 0$, the exact solution increases as t_n increases. The numerical solution must also increase with t_n. Thus, we are concerned with the relative accuracy, to a fixed number of significant figures, than with the absolute accuracy, to a fixed number of decimal places. Here, the relative stability is an important condition. This is ensured if the numerical solution does not increase faster than the true solution. For Euler's method we have

$$|1 + \lambda h| \leqslant e^{\lambda h}, \; \lambda > 0$$

which shows that the method is always relatively stable.

2.5.1 Fourth order Runge-Kutta method

We apply the classical fourth order Runge-Kutta method to Equation (2.45) and get

$$
\begin{aligned}
K_1 &= hf(t_n, y_n) \\
&= \lambda h\, y_n \\
K_2 &= hf(t_n+\tfrac{1}{2}h,\, y_n+\tfrac{1}{2}K_1) \\
&= \lambda h\,(y_n+\tfrac{1}{2}\lambda h\, y_n) \\
&= [\lambda h+\tfrac{1}{2}(\lambda h)^2]\, y_n \\
K_3 &= hf(t_n+\tfrac{1}{2}h,\, y_n+\tfrac{1}{2}K_2) \\
&= \lambda h\,(y_n+\tfrac{1}{2}(\lambda h+\tfrac{1}{2}(\lambda h)^2)\, y_n) \\
&= [\lambda h+\tfrac{1}{2}(\lambda h)^2+\tfrac{1}{4}(\lambda h)^3]\, y_n \\
K_4 &= hf(t_n+h,\, y_n+K_3) \\
&= \lambda h\,[y_n+(\lambda h+\tfrac{1}{2}(\lambda h)^2+\tfrac{1}{4}(\lambda h)^3)\, y_n] \\
&= [\lambda h+(\lambda h)^2+\tfrac{1}{2}(\lambda h)^3+\tfrac{1}{4}(\lambda h)^4]\, y_n \\
y_{n+1} &= y_n+\tfrac{1}{6}(K_1+2K_2+2K_3+K_4) \\
&= y_n+\tfrac{1}{6}(\lambda h)\, y_n+\tfrac{2}{6}(\lambda h+\tfrac{1}{2}(\lambda h)^2)\, y_n \\
&\quad +\tfrac{2}{6}(\lambda h+\tfrac{1}{2}(\lambda h)^2+\tfrac{1}{4}(\lambda h)^3)\, y_n \\
&\quad +\tfrac{1}{6}(\lambda h+(\lambda h)^2+\tfrac{1}{2}(\lambda h)^3+\tfrac{1}{4}(\lambda h)^4)\, y_n \\
&= [1+\lambda h+\tfrac{1}{2}(\lambda h)^2+\tfrac{1}{6}(\lambda h)^3+\tfrac{1}{24}(h\lambda)^4]\, y_n
\end{aligned}
$$

Thus, the *growth factor* for the fourth order method is

$$E(\lambda h)=1+\lambda h+\frac{(\lambda h)^2}{2!}+\frac{(\lambda h)^3}{3!}+\frac{(\lambda h)^4}{4!}$$

whereas the growth factor of the exact solution is $e^{\lambda h}$. If $\lambda h > 0$, then $e^{\lambda h} \geqslant E(\lambda h)$; so the fourth order Runge-Kutta method is always relatively stable. However, if $\lambda h < 0$, then to find the interval of absolute stability we construct the following table:

λh	0	−1	−2	−2.2	−2.6	−3.0
$E(\lambda h)$	1	0.3750	0.3330	0.4212	0.7547	1.375

The graph of $E(\lambda h)$ and $e^{\lambda h}$ for various order Runge-Kutta methods is shown in Figure 2.3. We notice from this graph that for $\lambda < 0$ the fourth order Runge-Kutta method first fails to be relatively stable, and then to be absolutely stable. The interval of absolute stability is $-2.78 < \lambda h < 0$.

2.5.2 Euler extrapolation method

The first column of the Y-scheme for the Euler extrapolation method for (2.48) is given by

$$Y_0^{(k)}=\left(1+\frac{\lambda h_0}{2^k}\right)^{2^k} y_n$$

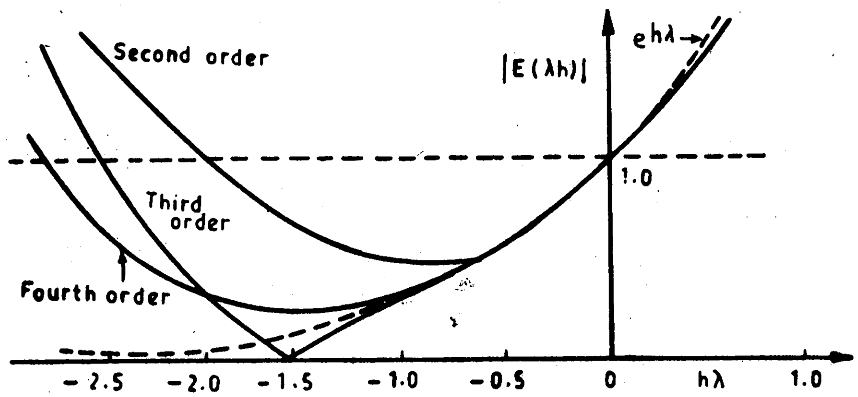

Fig. 2.3 Stability of Runge-Kutta method

and the other columns can be generated from the relation (2.40),

$$Y_m^{(k)} = \frac{2^m\, Y_{m-1}^{(k+1)} - Y_{m-1}^{(k)}}{2^m - 1} \tag{2.49}$$

$$= \sum_{j=0}^{m} c_{m,\,m-j}\, Y_0^{(k+j)}$$

$$= \left[\sum_{j=0}^{m} c_{m,\,m-j} \left(1 + \frac{\lambda h_0}{2^{k+j}}\right)^{2^{k+j}} \right] y_n$$

where
$$c_{m,m-j} = \frac{2^m\, c_{m-1,\,m-j} - c_{m-1,\,m-1-j}}{2^m - 1}$$

$$c_{m-1,m} = c_{m-1,\,-1} = 0$$

If for some $k = K$ and $m = M$, the extrapolated value $Y_M^{(K)}$ is taken as y_{n+1} then we can write (2.49) as

$$y_{n+1} = E\,(\lambda h_0, K, M)\, y_n$$

where
$$E\,(\lambda h_0, K, M) = \sum_{j=0}^{M} c_{M,M-j} \left(1 + \frac{\lambda h_0}{2^{K+j}}\right)^{2^{k+j}}$$

Thus the Euler extrapolation method is absolutely stable if

$$|\, E\,(\lambda h_0, K, M)\,| < 1$$

In order to find the interval of absolute stability for various values of K and M, we determine the values of λh_0 corresponding to K and M such that $|\, E\,(\lambda h_0, K, M)\,| = 1$. The principal diagonal of the Y-scheme converges faster than any other diagonal or column and so we determine the interval of absolute stability for $K = 0$ and various values of M. We have for $M = 5$

$$+\frac{1}{6}\left(c_i^3 h^3 f_{ttt}+3c_i^2 h^2 (a_{i1}K_1+a_{i2}K_2) f_{tty}\right.$$
$$+3c_i h (a_{i1}K_1+a_{i2}K_2)^2 f_{tyy}$$
$$\left.+(a_{i1}K_1+a_{i2}K_2)^3 f_{yyy}\right)+\ldots],\ i = 1, 2 \qquad (2.57)$$

Equations (2.57) are implicit and we cannot easily obtain the explicit expressions for K_1 and K_2. In order to determine K_1 and K_2 explicitly, we assume the following form

$$K_i = hA_i+h^2B_i+h^3C_i+h^4D_i+\ldots,\ i = 1, 2 \qquad (2.58)$$

where A_i, B_i, C_i and D_i are unknowns to be determined. Substituting for K_1 and K_2 from (2.58) into (2.57) and on equating powers of h, we obtain

$$A_i = f_n$$
$$B_i = c_i f_t+(a_{i1} A_1+a_{i2} A_2) f_y$$
$$C_i = (a_{i1} B_1+a_{i2} B_2) f_y+\frac{1}{2} c_i^2 f_{tt}+c_i (a_{i1} A_1+a_{i2} A_2) f_{ty}$$
$$+\frac{1}{2}(a_{i1} A_1+a_{i2} A_2)^2 f_{yy}$$
$$D_i = (a_{i1} C_1+a_{i2} C_2) f_y+c_i (a_{i1} B_1+a_{i2} B_2) f_{ty}$$
$$+(a_{i1} A_1+a_{i2} A_2)(a_{i1} B_1+a_{i2} B_2) f_{yy}$$
$$+\frac{1}{6} c_i^3 f_{ttt}+\frac{1}{2} c_i^2 (a_{i1} A_1+a_{i2} A_2) f_{tty}$$
$$+\frac{1}{2} c_i (a_{i1} A_1+a_{i2} A_2)^2 f_{tyy}$$
$$+\frac{1}{6}(a_{i1} A_1+a_{i2} A_2)^3 f_{yyy},\ i = 1, 2 \qquad (2.59)$$

Using (2.55) into (2.59), we get

$$A_i = f_n$$
$$B_i = c_i Df$$
$$C_i = (a_{i1} c_1+a_{i2} c_2) f_y Df+\frac{1}{2} c_i^2 D^2 f$$
$$D_i = [a_{i1} (a_{11} c_1+a_{12} c_2)+a_{i2} (a_{21} c_1+a_{22} c_2)] f_y^2 Df$$
$$+c_i (a_{i1} c_1+a_{i2} c_2) Df\, Df_y$$
$$+\frac{1}{2}(a_{i1} c_1^2+a_{i2} c_2^2) f_y D^2 f+\frac{1}{6} c_i^3 D^3 f,\ i = 1, 2 \qquad (2.60)$$

Equation (2.54) with the help of (2.58) may be written as

$$y_{n+1} = y_n+h (w_1 A_1+w_2 A_2)+h^2 (w_1 B_1+w_2 B_2)$$
$$+h^3 (w_1 C_1+w_2 C_2)+h^4 (w_1 D_1+w_2 D_2)+\ldots \qquad (2.61)$$

where A_i, B_i, C_i and D_i are given by (2.60). Comparing (2.61) with (2.56) and equating the powers of h, we can obtain implicit Runge-Kutta methods of various orders.

2.6.1 Second order method

Equating the coefficients of h and h^2, we get the following equations

$$w_1+w_2 = 1$$

$$w_1 c_1+w_2 c_2 = \frac{1}{2}$$

where $\qquad c_1 = a_{11}+a_{12},\ c_2 = a_{21}+a_{22}.$

There are now four arbitrary parameters to be prescribed. If we neglect K_2, i.e. if we choose $a_{21} = a_{22} = a_{12} = 0$, $w_2 = 0$ then on solving the above equations, we find

$$c_1 = \frac{1}{2},\ w_1 = 1$$

The second order implicit Runge-Kutta method with $v = 1$ is obtained

$$K_1 = hf\left(t_n+\frac{1}{2}h,\ y_n+\frac{1}{2}K_1\right)$$

$$y_{n+1} = y_n+K_1 \tag{2.62}$$

Applying (2.62) to $y' = \lambda y$, we have

$$y_{n+1} = \frac{1+\lambda h/2}{1-\lambda h/2}\, y_n$$

which gives a numerical method based on second order rational approximation to $e^{\lambda h}$ and it has stability interval $(-\infty, 0)$, $\lambda < 0$.

2.6.2 Third order method

Here we have the following system of equations

$$w_1+w_2 = 1$$

$$w_1c_1+w_2c_2 = \frac{1}{2}$$

$$w_1\,(a_{11}\,c_1+a_{12}\,c_2)+w_2\,(a_{21}\,c_1+a_{22}\,c_2) = \frac{1}{6}$$

$$w_1\,c_1^2+w_2\,c_2^2 = \frac{1}{3}$$

and $\qquad c_1 = a_{11}+a_{12},\ c_2 = a_{21}+a_{22} \qquad (2.63)$

The two arbitrary parameters can be chosen on the basis that either K_1 is explicit or K_2 is explicit. If we want K_1 to be explicit then we choose

$$a_{11} = a_{12} = 0$$

On solving (2.63), we get

$$c_1 = 0,\quad c_2 = \frac{2}{3},\ a_{21} = a_{22} = \frac{1}{3}$$

$$w_1 = \frac{1}{4},\ w_2 = \frac{3}{4}$$

If in (2.67), we choose $c_1=0$ or $c_v=1$, then (2.67) becomes *Radau's quadrature formula* and the implicit method has order $2v-1$. Using the condition (2.52), we have

(i) $c_1=0, a_{11}=a_{12}\cdots=a_{1v}=0$ and $c_2, c_3,\ldots, c_v$ are arbitrary

The arbitrary parameters $c_2, c_3,\cdots, c_v$ are the roots of the polynomial

$$\frac{d^{j-1}}{dc^{j-1}}[c^j(1-c)^{j-1}]=0,\ j=2, 3,\cdots, v \tag{2.70}$$

(ii) $c_v=1, a_{1v}=a_{2v}\cdots=a_{vv}=0$

The parameters $c_1, c_2,\ldots, c_{v-1}$ are arbitrary and can be chosen as the roots of the polynomial

$$\frac{d^{v-1}}{dc^{v-1}}[c^{v-1}(1-c)^v]=0 \tag{2.71}$$

Similarly, if we take $c_1=0, c_v=1$, then (2.67) becomes *Lobatto's quadrature formula* and the implicit method has order $2v-2$. In view of the condition (2.52), we get

$$c_1=0;\ a_{11}=a_{12}\cdots=a_{1v}=0$$

$$c_v=1;\ a_{1v}=a_{2v}\cdots=a_{vv}=0$$

and $c_2, c_3\ldots c_{v-1}$ are given by the roots of the polynomial

$$\frac{d^{v-2}}{dc^{v-2}}[c^{v-1}(1-c)^{v-1}]=0 \tag{2.72}$$

We now list a few high order methods.

Fifth order methods

0	0	0	0
$(6-\sqrt{6})/10$	$(9+\sqrt{6})/75$	$(24+\sqrt{6})/120$	$(168-73\sqrt{6})/600$
$(6+\sqrt{6})/10$	$(9-\sqrt{6})/75$	$(168+73\sqrt{6})/600$	$(24-\sqrt{6})/120$
	$1/9$	$(16+\sqrt{6})/36$	$(16-\sqrt{6})/36$

Runge-Kutta-Butcher method with Radau nodes

$(4-\sqrt{6})/10$	$(24-\sqrt{6})/120$	$(24-11\sqrt{6})/120$	0
$(4+\sqrt{6})/10$	$(24+11\sqrt{6})/120$	$(24+\sqrt{6})/120$	0
1	$(6-\sqrt{6})/12$	$(6+\sqrt{6})/12$	0
	$(16-\sqrt{6})/36$	$(16+\sqrt{6})/36$	$1/9$

Runge-Kutta-Butcher method with Radau nodes

Sixth order methods

$(5-\sqrt{15})/10$	$5/36$	$(10-3\sqrt{15})/45$	$(25-6\sqrt{15})/180$
$1/2$	$(10+3\sqrt{15})/72$	$2/9$	$(10-3\sqrt{15})/72$
$(5+\sqrt{15})/10$	$(25+6\sqrt{15})/180$	$(10+3\sqrt{15})/45$	$5/36$
	$5/18$	$8/18$	$5/18$

Runge-Kutta-Butcher method with Gaussian nodes

0	0	0	0	0
$(5-\sqrt{5})/10$	$(5+\sqrt{5})/60$	$1/6$	$(15-7\sqrt{5})/60$	0
$(5+\sqrt{5})/10$	$(5-\sqrt{5})/60$	$(15+7\sqrt{5})/60$	$1/6$	0
1	$1/6$	$(5-\sqrt{5})/12$	$(5+\sqrt{5})/12$	0
	$1/12$	$5/12$	$5/12$	$1/12$

Runge-Kutta-Butcher method with Lobatto nodes

Finally, the implicit Runge-Kutta methods have these advantages: They have large stability interval, and high order for the number of K_i's or the function evaluations. A disadvantage of the methods is that they require a system of linear or nonlinear equations depending on f, to be solved at each step.

2.7 OBRECHKOFF METHODS

The Taylor series method of order p can be obtained easily if it is possible to find the second and higher order derivatives from the given differential equation. The method is explicit and gives a pth degree polynomial approximation to $e^{\lambda h}$ when it is applied to the differential equation $y'=\lambda y$, $y(t_0)=y_0$. The interval of absolute stability is finite.

We shall now develop implicit single step method based on first p derivatives of $y(t)$ at t_n and t_{n+1}. The method has maximum order $2p$ and it is absolutely stable on $(-\infty, 0)$.

The general method is defined by

$$y_{n+1}=y_n+\sum_{i=1}^{q} a_i\, h^i\, y_{n+1}^{(i)}+\sum_{i=1}^{p} b_i\, h^i\, y_n^{(i)} \tag{2.73}$$

where a_i and b_i are arbitrary. The true value $y(t_n)$ will satisfy

$$T_n=y(t_{n+1})-y(t_n)-\sum_{i=1}^{q} a_i\, h^i\, y^{(i)}(t_{n+1})-\sum_{i=1}^{p} b_i\, h^i\, y^{(i)}(t_n) \tag{2.74}$$

Thus the method (2.75) becomes

$$y_{n+1} = y_n + \frac{2}{3} h y'_{n+1} - \frac{1}{6} h^2 y''_{n+1} + \frac{1}{3} h y'_n \tag{2.85}$$

The principal root of the characteristic equation of method (2.85) is given by

$$\xi = \frac{1 + \frac{1}{3}\lambda h}{1 - \frac{2}{3}\lambda h + \frac{1}{6}\lambda^2 h^2}$$

and it is plotted in Figure 2.6.

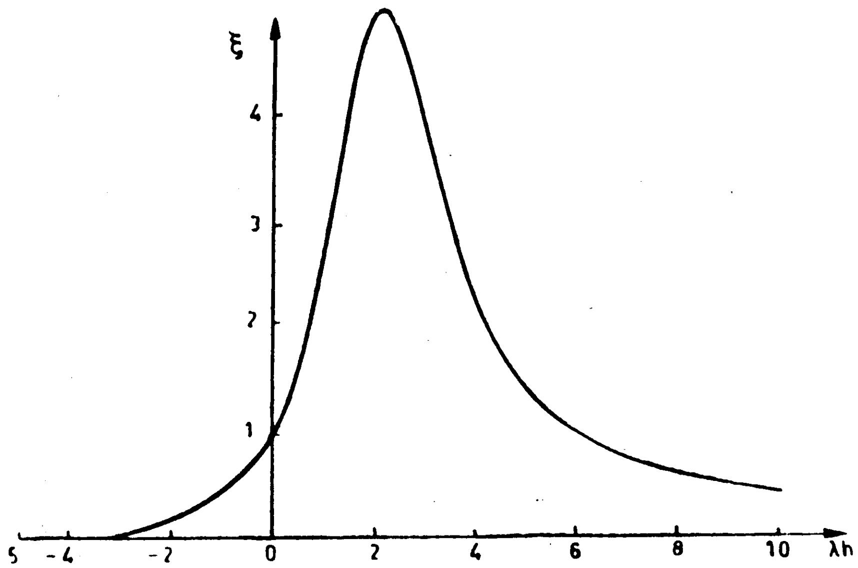

Fig. 2.6 Principal root of the third order method

We find that the method (2.85) is not only absolutely stable in the interval $(-\infty, 0)$ but it is also stable for all positive $\lambda h \geq 6$. Keeping a_1 arbitrary, we obtain from (2.84)

$$b_1 = 1 - a_1$$

$$a_2 = \frac{1}{6} - \frac{1}{2} a_1$$

$$b_2 = \frac{1}{3} - \frac{1}{2} a_1$$

Substituting in (2.75), we get

$$y_{n+1} = y_n + h\, a_1\, y'_{n+1} + h^2 \left(\frac{1}{6} - \frac{1}{2} a_1 \right) y''_{n+1} + h(1 - a_1)\, y'_n + h^2 \left(\frac{1}{3} - \frac{1}{2} a_1 \right) y''_n \tag{2.86}$$

a third order method with one arbitrary parameter. The principal root is found to be

$$\xi = \frac{1+\lambda h(1-a_1)+\lambda^2 h^2\left(\frac{1}{3}-\frac{1}{2}a_1\right)}{1-\lambda h a_1-\lambda^2 h^2\left(\frac{1}{6}-\frac{1}{2}a_1\right)}$$

The value $a_1 = 1+\sqrt{3}/3$ gives a third order method which has optimal stability.

2.7.3 Fourth order method

If, in addition to (2.84), we take

$$\frac{1}{6}a_1+\frac{1}{2}a_2=\frac{1}{24} \qquad (2.87)$$

then we obtain a fourth order method as

$$y_{n+1} = y_n+\frac{h}{2}(y'_{n+1}+y'_n)+\frac{h^2}{12}(-y''_{n+1}+y''_n) \qquad (2.88)$$

The method (2.88) is absolutely stable on $(-\infty, 0)$.

Alternatively, we may write (2.73) in the form

$$y_{n+1} = P_{p,q}(hD)\,y_n \qquad (2.89)$$

where
$$P_{p,q}(hD) = P_p(hD)/Q_q(hD)$$
$$P_p(hD) = 1+\sum_{i=1}^{p} b_i(hD)^i$$

and
$$Q_q(hD) = 1-\sum_{i=1}^{q} a_i(hD)^i$$

Equation (2.89) represents an approximation to the equation

$$y(t_{n+1}) = e^{hD}\,y(t_n)$$

The function $P_{p,q}(hD)$ is a rational approximation to e^{hD} Table 2.4 contains approximations of e^{hD}. Thus, depending on the values of p and q we obtain the following cases:

(i) $q = 0$, we get $b_i = 1/i!$ and (2.73) becomes the Taylor series method of order p.

(ii) $p = 0$, we obtain $a_i = (-1)^{i+1}\,1/i!$ and (2.73) becomes the backward Taylor series method of order q which is absolutely stable on $(-\infty, 0)$.

(iii) $p = q$, we find that $a_i = (-1)^{i+1} b_i$ and $b_i = \dfrac{p!\,(2p-i)!}{(2p)!\,(p-i)!\,i!}$

The method (2.73) becomes

$$y_{n+1} = y_n+\frac{p!}{(2p)!}\sum_{i=1}^{p}\frac{(2p-i)!}{(p-i)!\,i!}h^i\left[(-1)^{i-1}y^{(i)}_{n+1}+y^{(i)}_n\right]$$

and it has order $2p$. The interval of absolute stability is $(-\infty, 0)$.

2.8.2 Runge-Kutta methods

The classical fourth order Runge-Kutta formula becomes

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{1}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4) \tag{2.93}$$

where $\mathbf{K}_1 = \begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix}, \mathbf{K}_3 = \begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix}, \mathbf{K}_4 = \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}$

and $K_{j1} = hf_j(t_i, y_{1,i}, y_{2,i}, \ldots, y_{n,i})$

$$K_{j2} = hf_j\left(t_i + \frac{1}{2}h, y_{1,i} + \frac{1}{2}K_{11}, y_{2,i} + \frac{1}{2}K_{21}, \ldots, y_{n,i} + \frac{1}{2}K_{n1}\right)$$

$$K_{j3} = hf_j\left(t_i + \frac{1}{2}h, y_{1,i} + \frac{1}{2}K_{12}, y_{2,i} + \frac{1}{2}K_{22}, \ldots, y_{n,i} + \frac{1}{2}K_{n2}\right)$$

$$K_{j4} = hf_j(t_i + h, y_{1,i} + K_{13}, y_{2,i} + K_{23}, \ldots, y_{n,i} + K_{n3}), j = 1, 2, \ldots, n$$

In an explicit form (2.93) may be expressed as

$$\begin{bmatrix} y_{1,i+1} \\ y_{2,i+1} \\ \vdots \\ y_{n,i+1} \end{bmatrix} = \begin{bmatrix} y_{1,i} \\ y_{2,i} \\ \vdots \\ y_{n,i} \end{bmatrix} + \frac{1}{6}\left\{\begin{bmatrix} K_{11} \\ K_{21} \\ \vdots \\ K_{n1} \end{bmatrix} + 2\begin{bmatrix} K_{12} \\ K_{22} \\ \vdots \\ K_{n2} \end{bmatrix} + 2\begin{bmatrix} K_{13} \\ K_{23} \\ \vdots \\ K_{n3} \end{bmatrix} + \begin{bmatrix} K_{14} \\ K_{24} \\ \vdots \\ K_{n4} \end{bmatrix}\right\}$$

Example 2.4 Solve the initial value problem

$$x' = y, \quad x(0) = 0$$
$$y' = -x, \ y(0) = 1, \ t \in [0, 1]$$

by second order Runge-Kutta method with $h = 0.1$.

For $n = 0$

$t_0 = 0, x_0 = 0, y_0 = 1$

$K_{11} = hf_1(t_0, x_0, y_0) = .1(1) = .1$

$K_{21} = hf_2(t_0, x_0, y_0) = .1(0) = 0$

$K_{12} = hf_1(t_0+h, x_0+K_{11}, y_0+K_{21}) = .1(1+0) = .1$

$K_{22} = hf_2(t_0+h, x_0+K_{11}, y_0+K_{21}) = .1(0-.1) = -.01$

$$x_1 = x_0 + \frac{1}{2}(K_{11}+K_{12}) = 0 + \frac{1}{2}(.1+.1) = .1$$

$$y_1 = y_0 + \frac{1}{2}(K_{21}+K_{22}) = 1 + \frac{1}{2}(0-.01) = 1-.005 = .995$$

For $n = 1$

$t_1 = .1, x_1 = .1, y_1 = .995$

$K_{11} = hf_1(t_1, x_1, y_1) = .1(.995) = .0995$

$K_{21} = hf_2(t_1, x_1, y_1) = .1(-.1) = -.01$

$K_{12} = hf_1(t_1+h, x_1+K_{11}, y_1+K_{21}) = .1(.995-.01)$
$= .0985$

$$K_{22} = h f_2 (t_1+h, x_1+K_{11}, y_1+K_{21}) = .1 [-(.1+.0995)]$$

$$= -.01995$$

$$x_2 = x_1+\frac{1}{2} (K_{11}+K_{12})$$

$$= .1 +\frac{1}{2} (.0995+.0985)$$

$$= .1990$$

$$y_2 = y_1+\frac{1}{2} (K_{21}+K_{22})$$

$$= .995+\frac{1}{2} (-.01-.01995)$$

$$= .980025.$$

The exact solution is given by

$$x(t) = \sin t, \; y(t) = \cos t$$

The computed solution is listed in Table 2.5.

TABLE 2.5 SOLUTION OF $x' = y$, $y' = -x$, $x(0) = 0$, $y(0) = 1$ BY THE SECOND ORDER RUNGE-KUTTA METHOD WITH $h = 0.1$

t_n	x_n	y_n	$x(t_n)$	$y(t_n)$
0	0	1	0	1
0.1	0.1	0.995	0.099833	0.995005
0.2	0.1990	0.980025	0.198669	0.980067
0.3	0.296008	0.955225	0.295520	0.955336
0.4	0.390050	0.920848	0.389418	0.921061
0.5	0.480185	0.877239	0.479426	0.877583
0.6	0.565507	0.824834	0.564642	0.825336
0.7	0.645163	0.764159	0.644218	0.764842
0.8	0.718353	0.695822	0.717356	0.696707
0.9	0.784344	0.620508	0.783327	0.621610
1.0	0.842473	0.538971	0.841471	0.540302

2.8.3 Stability analysis

The stability of the numerical methods for the system of first order differential equations is discussed by applying the numerical methods to the homogeneous locally linearized form of the equation (2.90). Assuming that the functions f_i have continuous partial derivatives $\frac{\partial f_i}{\partial y_j} = a_{ij}$ and **A** denotes the $n \times n$ matrix $[a_{ij}]$, we may to terms of the first order write (2.90) as

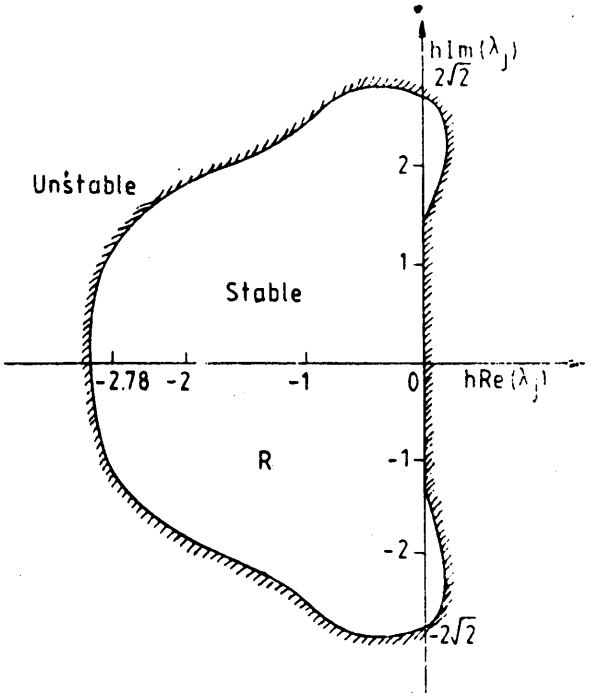

Fig. 2.7 Stability region for the fourth order Runge-Kutta method.

2.8.4 Stiff system of differential equations

There are many physical problems which lead to a system of ordinary differential equations with a property given by the following definition.

DEFINITION 2.4 A system of ordinary differential equations (2.90) is said to be stiff if the eigenvalues of the *Jacobian* matrix $\left[\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right]$ at every point of have negative real parts and differ greatly in magnitude.

We now study the main difficulties associated with the numerical solution of the stiff differential equations by applying the fourth order Runge-Kutta method to (2.94) when it is a stiff system, i.e., the eigenvalues λ_j of the matrix **A** satisfy the conditions:

(i) Real $\lambda_j < 0, j = 1, 2, \ldots, n$

(ii) $\max\limits_j$ | Real λ_j | $\gg \min\limits_j$ | Real λ_j |, $j = 1, 2, \ldots$

If λ_j is an eigenvalue of **A** whose real part is large in magnitude and $\bar{y}_j(t)$ represents, the component of the corresponding numerical solution then using (2.104) with $p = 4$, the following relationship is obtained

$$\bar{y}_j(t_{i+1}) = y_{j,i+1} = \left[1 + h\lambda_j + \frac{(h\lambda_j)^2}{2!} + \frac{(h\lambda_j)^3}{3!} + \frac{(h\lambda_j)^4}{4!}\right] y_{j,i}, \quad i = 0, 1, 2, \ldots \tag{2.105}$$

Initially we require $\bar{y}_j(t)$ to be accurate so we expect to keep $| h\lambda_j |$ small, but when $| \bar{y}_j(t) |$ has become negligible related to $y(t)$ it is unnecessary to require accuracy and we need only to ensure that $| \bar{y}_j(t) |$ does not grow. It has already been shown that to keep $| y_{j,i+1} | < | y_{j,i} |$ it is necessary that $| h\lambda_j | < 2.78$, approximately. Thus, the main difficulty encountered in solving stiff equations is that even though the component of the true solution corresponding to λ_j, soon becomes negligible, the restriction on step size imposed by the stability requires that $| h\lambda_j |$ remain small throughout the range of integration. Therefore, a numerical method must have very strong stability properties if it is to be efficient.

DEFINITION 2.5 A numerical method of the form (2.91) is called A-stable in the sense of *Dahlquist* if the region of stability associated with the method contains the open left-half-plane.

The fourth order Runge-Kutta method is not A-stable because it has finite region of stability on the left half-plane. The fully implicit Runge-Kutta-Butcher method (2.66) and other similar methods are A-stable. The *Obrechkoff* methods (2.89) are also A-stable. If we use the fourth order Runge-Kutta method to solve the stiff system of differential equations then we must limit the step size to a small value of the order of the reciprocal of the magnitude of the real part of the largest eigenvalue. Alternatively, if an A-stable implicit method is applied to a nonlinear system (2.90) it defines the value $y(t)$ at t_{n+1} of the numerical solution implicitly by the nonlinear algebraic system

$$y = g(y) \tag{2.106}$$

The stability requirement does no longer restrict the choice of step size. However we must solve the system (2.106) at each step by some iterative method. The convergence requirements for the iterative solution of (2.106) places restrictions on the largest step size that can be used. Thus, to solve the stiff differential system (2.90) we need not only the numerical methods with strong stability condition but also the accurate iterative methods for solving nonlinear algebraic system (2.106).

2.9 HIGHER ORDER DIFFERENTIAL EQUATIONS

The higher order equations can be solved by considering an equivalent system of first order equations. However, it is also possible to develop direct singlestep methods to solve higher order equations.

2.9.1 Runge-Kutta methods

Let us study the Runge-Kutta methods for a general second order equation

$$y'' = f(t, y, y'), \; t \in [t_0, b] \tag{2.107}$$

with the initial conditions

$$y_{n+1} = y_n + h\, y'_n + \frac{1}{2}(K_1+K_2)$$

$$y'_{n+1} = y'_n + \frac{1}{2h}(K_1+3K_2)$$

The Runge-Kutta method using four K's is given by

$$K_1 = \frac{h^2}{2} f(t_n, y_n, y'_n)$$

$$K_2 = \frac{h^2}{2} f\left(t_n+\frac{h}{2},\, y_n+\frac{1}{2}hy'_n+\frac{1}{4}K_1,\, y'_n+\frac{1}{h}K_1\right)$$

$$K_3 = \frac{h^2}{2} f\left(t_n+\frac{h}{2},\, y_n+\frac{1}{2}hy'_n+\frac{1}{4}K_1,\, y'_n+\frac{1}{h}K_2\right)$$

$$K_4 = \frac{h^2}{2} f\left(t_n+h,\, y_n+hy'_n+K_3,\, y'_n+\frac{2}{h}K_3\right)$$

$$y_{n+1} = y_n+hy'_n+\frac{1}{3}(K_1+K_2+K_3)$$

$$y'_{n+1} = y'_n+\frac{1}{3h}(K_1+2K_2+2K_3+K_4) \tag{2.116}$$

If the function f is independent of y', then we can construct a Runge-Kutta method in which the local truncation error in y and y' is $0\,(h^4)$. Here we get

$$W_1+W_2 = 1 \qquad W'_1+W'_2 = 2$$

$$a_2W_2 = \frac{1}{3} \qquad W'_2a_2 = 1$$

$$W'_2a_2^2 = \frac{2}{3}$$

$$W'_2a_{21} = \frac{2}{3}$$

which has the solution

$$a_2 = \frac{2}{3}, \qquad a_{21} = \frac{4}{9}$$

$$W_1 = W_2 = \frac{1}{2}, W'_2 = \frac{3}{2}, \quad W'_1 = \frac{1}{2}$$

Thus the Runge-Kutta method for the second order initial value problem

$$y'' = f(t, y)$$

$$y(t_0) = y_0,\ y'(t_0) = y'_0 \tag{2.117}$$

becomes

$$K_1 = \frac{h^2}{2!} f(t_n, y_n)$$

$$K_2 = \frac{h^2}{2!} f\left(t_n+\frac{2}{3}h,\, y_n+\frac{2}{3}h\,y'_n+\frac{4}{9}K_1\right) \tag{2.118}$$